

Digit Frequencies and Bernoulli Convolutions

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February 28, 2014

Abstract

It is well known that when β is a Pisot number, the corresponding Bernoulli convolution ν_β has Hausdorff dimension less than 1, i.e. that there exists a set A_β with $\nu_\beta(A_\beta) = 1$ and $\dim_H(A_\beta) < 1$. We show explicitly how to construct for each Pisot number β such a set A_β .

1 Introduction

Bernoulli convolutions are a simple class of fractal measures. Given $\beta \in (1, 2)$, we let $\pi_\beta : \{0, 1\}^\mathbb{N} \rightarrow I_\beta := \left[0, \frac{1}{\beta-1}\right]$ be given by

$$\pi_\beta(\underline{a}) = \sum_{i=1}^{\infty} a_i \beta^{-i}.$$

We let m be the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure on $\{0, 1\}^\mathbb{N}$ and define the Bernoulli convolution ν_β by

$$\nu_\beta = m \circ \pi_\beta^{-1}.$$

It is a major open problem to determine the values of β for which ν_β is absolutely continuous. Erdős proved in [7] that if $\beta \in (1, 2)$ is a Pisot number then the Bernoulli convolution ν_β is singular, while in [9] Garsia provided a countable number of values of β for which ν_β is absolutely continuous. Solomyak proved in [25] that for almost every $\beta \in (1, 2)$ the corresponding Bernoulli convolution is absolutely continuous, and very recently Shmerkin [22] has proved that the set of β for which ν_β is singular has zero Hausdorff dimension, yet for typical given β the problem of determining whether ν_β is absolutely continuous remains open.

Given that ν_β is totally singular when β is a Pisot number, it seems that one of the most fundamental questions associated to ν_β is to understand where it is supported, can one describe a set of zero Lebesgue measure but with full ν_β -measure? In this note we answer the above question by giving examples of sets A_β of Hausdorff dimension less than one but with $\nu_\beta(A_\beta) = 1$.

We do this first for the special case that β is equal to the golden mean. This case has been of particular interest, with extensive investigations into questions relating to entropy, Hausdorff dimension, Gibbs properties and multifractal analysis of the corresponding Bernoulli convolution, see for example [1, 12, 14, 15, 17, 24]. Our method here uses elementary calculations on digit frequencies for the greedy β -transformation.

In the final section we generalise our techniques to all Pisot numbers, this is more involved and requires us to further develop the theory of normalisation for β -expansions.

1.1 The Greedy β -transformation

In [21], Rényi introduced the greedy β -transformation $T_\beta : I_\beta \rightarrow I_\beta$ defined by

$$T_\beta(x) := \begin{cases} \beta x & x \in [0, \frac{1}{\beta}) \\ \beta x - 1 & x \in [\frac{1}{\beta}, \frac{1}{\beta-1}] \end{cases}.$$

The map T_β has $[0, 1]$ as an attractor, and $T_\beta(x)|_{[0,1]} = \beta x \pmod{1}$. In [18] it was proved that T_β preserves the absolutely continuous probability measure μ_β with density given by

$$\rho_\beta(x) = C_\beta \sum_{i=0}^{\infty} \frac{1}{\beta^i} \chi_{[0, T_\beta^i(1)]}(x)$$

where C_β is a normalising constant.

Given $x \in I_\beta$ we generate a sequence (x_n) known as the greedy β -expansion of x by iterating T_β . Letting $x_n = 0$ when $T_\beta^n(x) = \beta(T_\beta^{n-1}(x))$ and $x_n = 1$ when $T_\beta^n(x) = \beta(T_\beta^{n-1}(x)) - 1$, we have that the sequence (x_n) satisfies

$$\sum_{n=1}^{\infty} x_n \beta^{-n} = x.$$

We let

$$X_\beta := \overline{\left\{ (x_n) \in \{0, 1\}^{\mathbb{N}} : (x_n) \text{ is a greedy } \beta\text{-expansion of some } x \in \left[0, \frac{1}{\beta-1}\right] \right\}},$$

where \overline{X} denotes the closure of X . We define the left shift $\sigma : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ by $\sigma(x_n)_{n=1}^{\infty} = (x_{n+1})_{n=1}^{\infty}$. We further define the lexicographical ordering \prec on $\{0, 1\}^{\mathbb{N}}$ by declaring that $(x_n) \prec (y_n)$ if and only if $x_1 < y_1$ or if there exists $n \in \mathbb{N}$ with $x_1 \cdots x_{n-1} = y_1 \cdots y_{n-1}$ and $x_n < y_n$.

In [18], Parry characterised the set of β -expansions of $x \in I_\beta$ in terms of the orbit of the point 1 under T_β . For β equal to the golden mean we have that any sequence (x_n) without two consecutive occurrences of digit 1 is a greedy β -expansion of some $x \in [0, 1]$. Points $x \in [1, \frac{1}{\beta-1})$ have greedy expansions which start with m 1s for some $m \geq 2$, but thereafter will have no consecutive occurrence of digit 1.

Applying the ergodic theorem to $(I_\beta, \mu_\beta, T_\beta)$ with the characteristic function on $[\frac{1}{\beta}, \beta]$ gives the following theorem.

Theorem 1.1 (Parry [18]). *Let $\beta = \frac{1+\sqrt{5}}{2}$. For Lebesgue almost every $x \in I_\beta$ the frequency of the digit 1 in the greedy β -expansion of x is given by*

$$\alpha(1) := \mu \left[\frac{1}{\beta}, 1 \right] = \frac{1}{\beta^2 + 1} \approx 0.27639.$$

Indeed, it follows from Theorem 17 of [3] that, letting

$$A_\gamma = \{x \in I_\beta : \text{the greedy } \beta\text{-expansion of } x \text{ has digit 1 with frequency } \gamma\},$$

we have that $\dim_H(A_\gamma) < 1$ for each $\gamma \neq \alpha(1)$.

Simple analysis reveals that ν_β -almost every $x \in [0, 1]$ has β -expansion in which the digit 1 occurs with frequency different from $\alpha(1)$.

Theorem 1.2. *Let $\beta = \frac{1+\sqrt{5}}{2}$. For ν_β -almost every $x \in I_\beta$, the frequency of the digit 1 in the greedy β -expansion of x is equal to $\frac{5}{18} = 0.2\dot{7} > \alpha(1)$.*

We prove this theorem in the next section. An immediate corollary is that ν_β is singular with Hausdorff dimension

$$\dim_H(\nu_\beta) \leq \dim_H(A_{\frac{5}{18}}) < 1.$$

In the final section we generalise our result to all Pisot numbers. This involves building on techniques of Frougny [8] and Sidorov [23].

Given a natural number n , the ergodic theorem applied to the greedy β -transformation gives for each $a_1 \cdots a_n \in \{0, 1\}^n$ a frequency $P_{a_1 \cdots a_n}$ such that for Lebesgue almost every x , the word $a_1 \cdots a_n$ appears in the greedy β -expansion of x with frequency $P_{a_1 \cdots a_n}$.

Definition 1.1. *We call a set A a set of non-standard digit frequency if there exists a word $a_1 \cdots a_n$ and a number $Q_{a_1 \cdots a_n} \neq P_{a_1 \cdots a_n}$ such that the frequency of the occurrence of $a_1 \cdots a_n$ in the greedy β -expansion of x is equal to $Q_{a_1 \cdots a_n}$ for every $x \in A$.*

Sets of non-standard digit frequency have Hausdorff dimension less than one.

Theorem 1.3. *Let $\beta \in (1, 2)$ be a Pisot number. There exists a measure invariant under the β -transformation T_β with the same Hausdorff dimension as ν_β . Furthermore, there exist a finite number of sets $A_1 \cdots A_k$ of non-standard digit frequency such that ν_β is supported on the union of the sets A_k . For each β , the invariant measure and suitable sets $A_1 \cdots A_k$ can be explicitly described by an algorithm which terminates in finite time.*

The invariant measure described in the above theorem is a hidden Markov measure. A hidden Markov measure is an image of a Markov measure on a Markov shift under a one block factor map, which is just a map which concatenates some symbols in the alphabet of the Markov shift. Hidden Markov measures have been widely studied [16], and it seems likely that expressing Bernoulli convolutions as such might be useful. We discuss this further in the final section.

2 Normalising β -expansions for the golden mean.

In this section we provide a proof of Theorem 1.2. One could provide a shorter proof of this theorem by using the work of [24], but we prefer to keep this section simple and self-contained.

We define a map $P : \{0, 1\}^{\mathbb{N}} \rightarrow X_\beta$ by $P(a_n) = (x_n)$ where x_n is the greedy expansion of $\sum_{i=1}^{\infty} a_i \beta^{-i}$. This process is called normalising and has been studied, for example, in [4, 24]. Indeed, [24] uses the map P to provide a much more detailed description of the measure ν_β .

Lemma 2.1 (Sidorov, Vershik [24]). *Given $(a_n) \in \{0, 1\}^{\mathbb{N}}$, the sequence $P(a_n)$ can be found as the limit of the sequences obtained by iterating the following algorithm. First look for the first occurrence in (a_n) of the word 011. If such a word occurs, replace it with 100 and then repeat from the start, looking for the first occurrence of the word 011 in our new sequence.*

Proof. Because the golden mean satisfies the equation

$$\beta^{-n} = \beta^{-(n+1)} + \beta^{-(n+2)}$$

for any $n \in \mathbb{N}$ we see that the above algorithm produces another β -expansion of the same point. Furthermore, the sequence $P(a_n)$ will have no two consecutive occurrences of digit 1, except that it may start with word 1^m , and thus $\sigma^k P(a_n)$ does not contain two consecutive 1s for any $k \geq m$, giving that $P(a_n)$ is a greedy sequence. \square

We extend the definition of P to almost all two-sided sequences $(a_n) \in \{0, 1\}^{\mathbb{Z}}$ as follows. First we define P on finite words $a_1 \cdots a_n$ as in the above algorithm. For a sequence $(a_n)_{n=1}^{\infty}$ for which there exist infinitely many $n \in \mathbb{N}$ for which $a_n = a_{n+1} = 0$, we can rewrite $(a_n) = w_1 w_2 \cdots$ where w_i is a finite word which starts with 00 for each $i \geq 1$. Then we observe that $P(a_n) = P(w_1)P(w_2) \cdots$, see [24] for a proof. Finally, if $(a_n)_{n=-\infty}^{\infty}$ is a two sided sequence for which there exist infinitely many positive and negative n with $a_n = a_{n+1} = 0$, we can again write (a_n) as a concatenation of words w_i that start with 00 and define $P(a_n)$ to be the concatenation of the normalised words $P(w_i)$.

We also observe that if one has a two sided sequence $(x_n)_{n=-\infty}^{\infty}$ and one defines $y^+ = P((x_n)_{n=1}^{\infty})$ and $y^- = P((x_n)_{n=-\infty}^0)$ then $P((x_n)_{n=-\infty}^{\infty}) = P(y^- y^+)$.

Our strategy goes as follows. In order to find the expected asymptotic frequency of the digit 1 in the sequence $P(x^+)$ for a randomly chosen $x^+ = (x_n)_{n=1}^{\infty}$ we choose a random ‘past’ $x^- = (x_n)_{n=-\infty}^0$, where the negative digits are chosen independently from $\{0, 1\}$ with probability $\frac{1}{2}$, and instead normalise the two sided sequence $x^- x^+$. The two sided sequence $P(x^- x^+)$ agrees with $P(x^+)$ for all positive n for which there exists $0 < m < n$ with $x_m = x_{m+1} = 0$, and thus has the same limiting frequency of the digit 1 in the positive coordinates.

Since the probability that the n th term of $P(x^- x^+)$ is equal to 1 is independent of n , we just need to calculate the probability that $(P(x^- x^+))_0 = 1$ and then use the ergodic

theorem together with the shift map.

Now since $(\{0, 1\}^{\mathbb{N}}, m, \sigma)$ is ergodic, the ergodic theorem using the characteristic function on the set of sequences with $(P(x^-x^+))_0 = 1$ gives that the limiting frequency of the digit 1 in sequences $P(a_n)$ is equal to

$$m\{(a_n) \in \{0, 1\}^{\mathbb{N}} : (P(a_n))_0 = 1\}.$$

In the next section we calculate the measure of this set, we introduce the shorthand

$$\mathbb{P}(x_n = i) := m\{a \in \{0, 1\}^{\mathbb{N}} : (P(a))_n = i\}$$

and refer to this as the probability that x_n is equal to i .

2.1 Calculations

In this section we make some simple computations which prove Theorem 1.2.

Lemma 2.2. *Let $P(x^+) = (y_n)_{n=1}^{\infty}$. Then the probability that $y_1 = 1$ is equal to $\frac{2}{3}$, and the probability that $y_1 = y_2 = 1$ is equal to $\frac{1}{3}$.*

Proof. The cylinder sets $A_k = [(01)^k 1]$ and $B_k = [(01)^k 00]$ for $k \geq 0$ partition $\{0, 1\}^{\mathbb{N}}$. Then using that $P((01)^k 1) = 10^{2k}$ and $P((01)^k 00) = (01)^k 00$ we see that $y_1 = 1$ if and only if $x^+ \in A_k$ for some $k \geq 0$. Thus the probability that $y_1 = 1$ is given by

$$\mathbb{P}(y_1 = 1) = \sum_{k=0}^{\infty} \frac{1}{2} \cdot \frac{1}{4^k} = \frac{1}{2} \left(\frac{1}{1 - \frac{1}{4}} \right) = \frac{2}{3}$$

Similarly, we see that $y_1 = y_2 = 1$ if and only if x^+ belongs to the cylinder set $[1(01)^k 1]$ for some $k \geq 0$, which happens with probability

$$\mathbb{P}(y_1 = y_2 = 1) = \sum_{k=0}^{\infty} \frac{1}{4} \cdot \frac{1}{4^k} = \frac{1}{4} \left(\frac{1}{1 - \frac{1}{4}} \right) = \frac{1}{3}$$

□

Lemma 2.3. *Let $P(x^-) = (y_n)_{n=-\infty}^0$. Then the probability that $y_0 = 1$ is equal to $\frac{1}{3}$, and the probability that $y_{-1} = y_0 = 0$ is equal to $\frac{1}{2}$.*

Proof. The cylinder sets $A_k := {}_{-2k} [01^{2k}]_0$ and $B_k := {}_{-2k-1} [01^{2k+1}]_0$ for $k \geq 0$ partition the set $\{(x_n)_{n=-\infty}^0 : x_i \in \{0, 1\}\}$. We use the identities $P(01^{2k}) = (10)^k 0$ and $P(01^{2k+1}) = (10)^k 01$. Then we see that $y_0 = 1$ if and only if $x^- \in B_k$ for some $k \geq 0$. This happens with probability

$$\mathbb{P}(y_0 = 1) = \sum_{k=0}^{\infty} \frac{1}{4} \left(\frac{1}{4} \right)^k = \frac{1}{3}$$

Furthermore we see that $y_0 = y_{-1} = 0$ if and only if $x_{-2k} \cdots x_0 = 01^{2k}$ for some $k \geq 1$ or $x_{-2k-1} \cdots x_0 = 01^{2k}0$ for some $k \geq 0$. This happens with probability

$$\mathbb{P}(y_{-1} = y_0 = 0) = \left(\sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{1}{4} \right)^k \right) + \left(\sum_{k=0}^{\infty} \left(\frac{1}{4} \right)^{k+1} \right) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

□

Finally, we study $P(y^-y^+)$.

Lemma 2.4. *Letting $(x_n)_{n=-\infty}^{\infty} = P(y^-y^+)$ we have that $x_0 = 1$ if and only if one of the following two conditions holds.*

Case 1: $y_0 = 1, y_1 = 0$.

Case 2: $y_{-1} = y_0 = 0, y_1 = y_2 = 1$.

Thus the probability that $x_0 = 1$ is equal to $\frac{5}{18}$.

Proof. $P(y^-y^+) = (x_n)_{n=1}^{\infty}$. Since y^- and y^+ are already normalised, we see that if $y_1 = 0$ then the concatenation y^-y^+ does not contain consecutive occurrences of the digit 1 and hence is normalised, giving $x_0 = y_0$. Thus if $y_1 = 0$ then $x_0 = 1$ if and only if $y_0 = 1$.

Now suppose that $y_1 = 1$. If $y_0 = 1$ then, since y^- is normalised we must have that $y_{-1} = 0$. Then when we normalise y^-y^+ the 011 in position $y_{-1}y_0y_1$ is replaced by 100, giving that $x_0 = 0$.

We suppose that $y_1 = 1$ and $y_0 = 0$. If $y_2 = 0$ then y^-y^+ is normalised and $x_0 = 0$. Thus we assume that $y_2 = 1$. Then the first action of P is to turn the 011 in position $y_0y_1y_2$ into 100. If $y_{-1} = 1$ then subsequent substitutions of 011 with 100 will leave $x_n = 0$. However if $y_{-1} = 0$ then this sequence is normalised.

Adding the relevant probabilities computed earlier, we see that

$$\begin{aligned} \mathbb{P}(x_0 = 1) &= \mathbb{P}(y_0 = 1, y_1 = 0) + \mathbb{P}(y_{-1} = y_0 = 0, y_1 = y_2 = 1) \\ &= \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{5}{18} \end{aligned}$$

□

Then by the ergodic theorem, we observe that for m -almost every sequence $(a_n) \in \{0, 1\}^{\mathbb{N}}$, the sequence $P(a_n)$ has digit 1 with limiting frequency $\frac{5}{18}$. Since $\nu_\beta = m \circ \pi^{-1}$ it follows that ν_β is supported on the set $A_{\frac{5}{18}}$.

3 Normalisation for Other Pisot Numbers

In this section we study normalisation of β -expansions for general Pisot numbers $\beta \in (1, 2)$. We let P_β be the map which takes a sequence in $\{0, 1\}^\mathbb{N}$ to the corresponding greedy β -expansion.

We begin by recalling earlier work on normalisation of Frougny and Sidorov. Sidorov described a process of two-sided normalisation which works for all Pisot numbers β which satisfy a property known as weak finiteness, and Theorem 1.3 follows rather directly using this process whenever weak finiteness holds. It is conjectured that all Pisot numbers satisfy weak finiteness, see [23, 2] and the references therein, but despite a good deal of attention over the last fifteen years this conjecture remains unsolved. Our approach to normalisation does not require the weak finiteness property, and thus works for all Pisot numbers.

The following theorem of Frougny was phrased in terms of cellular automata but we prefer to avoid this notation here.

Theorem 3.1 (Frougny [8]). *There exists a finite alphabet \mathcal{A} , a one sided Markov shift $\Sigma^+ \subset \mathcal{A}^\mathbb{N}$, and maps $\phi_1 : \{0, 1\}^\mathbb{N} \rightarrow \Sigma^+$ and $\phi_2 : \Sigma^+ \rightarrow X_\beta$ satisfying that for each $\underline{a} \in \{0, 1\}^\mathbb{N}$ one has*

$$\phi_2 \circ \phi_1(\underline{a}) = P_\beta(\underline{a})$$

for m -almost every $\underline{a} \in \{0, 1\}^\mathbb{N}$.

We let G denote the graph whose vertices are the letters of \mathcal{A} and whose edges are the allowed transitions in the Markov shift Σ^+ . The map ϕ_1 takes the form of deciding an initial vertex $e \in \mathcal{A}$ and then labelling each edge of the graph G with a 0 or 1 in such a way that almost every sequence $\underline{a} \in \{0, 1\}^\mathbb{N}$ can be represented uniquely as the infinite path in G starting from e and following a sequence of edges whose labels correspond to the sequence \underline{a} . $\phi_1(\underline{a}) \in \Sigma$ is then defined to be the vertex sequence for this infinite path in G .

The map ϕ_2 starts with defining a map $\phi'_2 : \mathcal{A} \rightarrow \{0, 1\}$ and then extending this to a one block factor map from $\Sigma^+ \rightarrow X_\beta$ by letting $(\phi_2(\underline{a}))_{n=1}^\infty = (\phi'_2(a_n))_{n=1}^\infty$.

Frougny's method computes normalisation exactly, but because normalisation is not shift invariant it is difficult to use it for studying the ergodic properties of ν_β . Later, Sidorov constructed measures invariant under T_β which are equivalent to Bernoulli convolutions for certain Pisot numbers. One consequence of his result is the following.

Proposition 3.1 (Sidorov [23]). *Suppose that $\beta \in (1, 2)$ is a Pisot number satisfying the weak finiteness property. Then for almost every two sided sequence $\underline{a} \in \{0, 1\}^\mathbb{Z}$ there exists a unique two-sided infinite path in G such that for each $n \in \mathbb{Z}$ the edge traversed at time n has label a_n .*

Letting Σ be the two sided version of Σ^+ , one can extend the map ϕ_1 to let it be the map from $\{0, 1\}^\mathbb{Z}$ to Σ which takes $\underline{a} \in \{0, 1\}^\mathbb{Z}$ to the corresponding vertex sequence in G . $\phi_2 \circ \phi_1(\underline{a})$ corresponds to the two sided normalisation which Sidorov defined. One

can prove Theorem 1.3 for the special case that β -satisfies weak finiteness using this method.

We show that, even when there doesn't exist a unique path in G corresponding to $\underline{a} \in \{0, 1\}^{\mathbb{Z}}$, there exist a bounded number of paths. This allows us to prove Theorem 1.3. Our first result is the following

Proposition 3.2. *Let $\beta \in (1, 2)$ be a Pisot number. There exists a number $k \in \mathbb{N}$ such that for almost every sequence $\underline{a} \in \{0, 1\}^{\mathbb{Z}}$ there are exactly k two-sided infinite paths on G which traverse an edge labeled by a_n at each time $n \in \mathbb{Z}$.*

Of course, if the conjecture that every Pisot number satisfies weak finiteness is true then k will always be equal to one by the result of Sidorov.

Proof. Let $k \in \mathbb{N}$ be the smallest natural number such that there exists a word $w_1 \cdots w_n \in \{0, 1\}^n$ such that whenever we follow a sequence of edges labelled by $w_1 \cdots w_n$ in G we know that we are in one of at most k vertices, i_1, \dots, i_k . Such a finite number k must exist, since the vertices of G are labelled by \mathcal{A} , which is a finite set, and thus $k \leq |\mathcal{A}|$.

Now we suppose that there was a word $x_1 \cdots x_m$ such that from two different vertices $i_{j_1}, i_{j_2} \in \{i_1, \dots, i_k\}$ we could follow an edge pattern $x_1 \cdots x_m w_1 \cdots w_n$ and end up at the same vertex $i_{j_3} \in \{i_1, \dots, i_k\}$. Then there would be strictly fewer than k vertices of the graph which one could reach after following edge sequence $w_1 \cdots w_n x_1 \cdots x_m w_1 \cdots w_n$. This would contradict k being minimal. Thus, if we know which vertex of the graph we are at after following edge sequence $w_1 \cdots w_n x_1 \cdots x_m w_1 \cdots w_n$, we know where we were after just following the first $w_1 \cdots w_n$.

Now suppose that $\underline{a} \in \{0, 1\}^{\mathbb{Z}}$ have infinitely many positive and negative occurrences of the word $w_1 \cdots w_n$. Let us choose a position $i \in \{i_1, \dots, i_k\}$ in which to be after the first time we have $a_m \cdots a_{m+n-1} = w_1 \cdots w_n$ for $m \in \mathbb{N}$, the first positive time occurrence of $w_1 \cdots w_n \in \underline{a}$. By the argument of the previous paragraph, this uniquely defines which value of $\{i_1, \dots, i_k\}$ we lie in after the first negative occurrence of $w_1 \cdots w_n$, which in turn defines which value we lie in after the second negative occurrence of $w_1 \cdots w_n$.

Iterating this process, we see that given the word \underline{a} and a choice $i \in \{i_1, \dots, i_k\}$, there exists a unique sequence $\underline{b} \in \Sigma$ such that one can get from each b_n to b_{n+1} by following an edge labelled with a_n , and such that we pass through position i after the first positive occurrence of $w_1 \cdots w_n$. This defines a bijection $\phi'_1 : \{0, 1\}^{\mathbb{Z}} \times \{1, \dots, k\} \rightarrow \Sigma$ up to sets of m measure 0. \square

We let m_k be the measure on $\{i_1, \dots, i_k\}$ which gives mass $\frac{1}{k}$ to each letter. Recall that m is the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure on $\{0, 1\}^{\mathbb{Z}}$.

The map ϕ'_1 depends on the second coordinate i , which determines where we project to on G after the first occurrence of $w_1 \cdots w_n$ in the positive coordinates. We need to deal with the situation that, by shifting on $\{0, 1\}^{\mathbb{Z}}$, some occurrence of $w_1 \cdots w_n$ which was previously in positive coordinates has been moved to negative coordinates.

Given \underline{a} with $a_1 \cdots a_{2n+m} = w_1 \cdots w_n x_1 \cdots x_m w_1 \cdots w_n$, we let $S(\underline{a}, i) \in \mathcal{A}$ be the vertex of G that we would reach if we started at position i and then follow edges labelled by $x_1 \cdots x_m w_1 \cdots w_n$.

We define a map $T : \{0, 1\}^{\mathbb{Z}} \times \{i_1, \dots, i_k\} \rightarrow \{0, 1\}^{\mathbb{Z}} \times \{i_1, \dots, i_k\}$ by

$$T(\underline{a}, i) := \begin{cases} (\sigma(\underline{a}), i) & a_1 \cdots a_n \neq w_1 \cdots w_n \\ (\sigma(\underline{a}), S(\underline{a}, i)) & a_1 \cdots a_n = w_1 \cdots w_n \end{cases}.$$

Lemma 3.1. *The measure $m \times m_k$ on $\{0, 1\}^{\mathbb{Z}} \times \{i_1, \dots, i_k\}$ is invariant under T . Furthermore, the system $(\{0, 1\}^{\mathbb{Z}} \times \{i_1, \dots, i_k\}, T, m \times m_k)$ has at most k ergodic components.*

Proof. Invariance of the measure $m \times m_k$ is immediate. Now we suppose that B is invariant under T and has $(m \times m_k)(B) > 0$. We write

$$B = (B_1 \times \{i_1\}) \cup (B_2 \times \{i_2\}) \cup \cdots \cup (B_k \times \{i_k\})$$

Since the transformation T always shifts in the first coordinate, the set $B_1 \cup B_2 \cup \cdots \cup B_k \subset \{0, 1\}^{\mathbb{Z}}$ must have

$$m(B_1 \cup B_2 \cup \cdots \cup B_k) = 1,$$

since it is an invariant set of positive measure and the shift map is ergodic. Thus the set B invariant under T must have $(m \times m_k)$ -measure at least $\frac{1}{k}$. \square

Lemma 3.2. *The system (Σ, σ) is a factor of $(\{0, 1\}^{\mathbb{Z}} \times \{i_1, \dots, i_k\}, T)$. Furthermore the measure $m \times m_k$ on $\{0, 1\}^{\mathbb{Z}} \times \{i_1, \dots, i_k\}$ is mapped by $\phi_2 \circ \phi'_1$ to a shift invariant measure $\tilde{\mu}_2$ on the two sided β -shift \tilde{X}_β with at most k ergodic components $(\mu_2^1, \dots, \mu_2^k)$.*

Proof. We need to check that for $\underline{a} \in \mathbb{Z}, i \in \{i_1, \dots, i_k\}$ we have that

$$\phi'_1 \circ T(\underline{a}, i) = \sigma \circ \phi'_1(\underline{a}, i).$$

But we defined T precisely so that this relation holds, the only complication comes from our notion of the smallest $m \in \mathbb{N}$ such that $a_m \cdots a_{m+n-1} = w_1 \cdots w_n$. If $m > 1$, i.e. if $a_1 \cdots a_n \neq w_1 \cdots w_n$ we just shift \underline{a} and this corresponds to shifting $\phi'_1(\underline{a}, i)$. If $m = 1$ we need also to change i , but the appropriate change is defined by S . So we see that (Σ, σ) is a factor of $(\{0, 1\}^{\mathbb{Z}} \times \{i_1, \dots, i_k\}, T)$.

Now since shift invariance and ergodicity are preserved by passing to factor maps, and $m \times m_k$ is invariant with at most k -ergodic components, it follows that $\tilde{\mu}_2 := (m \times m_k) \circ \phi_1'^{-1}$ will be shift invariant with at most k ergodic components. \square

We let μ_2 be the restriction of $\tilde{\mu}_2$ to X_β , i.e. the restriction to positive coordinates.

For $i \in \{i_1, \dots, i_k\}$ let A_i be the set of sequences $\underline{a} \in \{0, 1\}^{\mathbb{N}}$ for which the limiting frequencies of occurrences of each word $x_1 \cdots x_n$ are equal to $\mu_2^i[x_1 \cdots x_n]$. We further define the measures μ_3^i as the projections of μ_2^i onto I_β by

$$\mu_3^i = \mu_2^i \circ \pi_\beta^{-1}.$$

Since π_β maps X_β bijectively onto $[0, 1]$, this just allows us to deal with $[0, 1]$ rather than symbolic space.

Now we are able to complete the proof of Theorem 1.3.

Proof. For almost every $\underline{a} \in \{0, 1\}^\mathbb{N}$ there exists $j \in \{i_1, \dots, i_k\}$ such that the m th terms of the sequences $P_\beta(\underline{a})$ and $(\phi_2 \circ \phi'_1(\underline{a}, i))$ agree for all m large enough. This follows because, following Frougny's algorithm, if we start in position e of our graph G and follow edge sequence given by $a_1 \dots a_m$, where $a_{m-n+1} \dots a_m$ are the first occurrence of word $w_1 \dots w_n$ in \underline{a} , then we will reach some position $j \in \{i_1 \dots i_k\}$. The rest of the sequence generated by P_β is the same as that of $(\phi_2 \circ \phi'_1(\underline{a}, j))$.

In particular, since the digit frequencies of a sequence do not depend on any finite number of digits, for almost every $\underline{a} \in \{0, 1\}^\mathbb{Z}$ we have that $P_\beta(\underline{a})$ will be the greedy expansion of a point in some A_i . Thus ν_β is supported on the union of the sets A_i .

In fact we can write $m \circ P_\beta$ in terms of weighted sums of the measures ν_2^i on cylinders $\phi_2 \circ \phi'_1([x_1 \dots x_m w_1 \dots w_n], i)$. Then ν_β is written in terms of weighted sums of measures μ_3^i . This gives that the Hausdorff dimension of ν_β coincides with that of the measures μ_3^i , as required.

□

4 Further Comments

There are a few of directions in which it seems natural to build on the ideas of this article.

Firstly, the the measures μ_3^i are hidden Markov measures. There is a wealth of literature studying the entropy of hidden Markov measures, and in some cases one has algorithms approximating the entropy which converge superexponentially [19]. These would lead to algorithms for computing the dimension of ν_β which converge superexponentially. For certain special cases, rapid algorithms for approximating the Hausdorff dimension of a Bernoulli convolution do exist [1, 6, 10], but these are somewhat specialised and it would be interesting to see whether, using hidden Markov measures, one could do something more general.

Similarly, there is much research into way that the entropy of a hidden Markov measure depends on the parameters defining the measure, and in some cases this dependence is analytic [11]. This could be used to study the dependence of the dimension of a biased Bernoulli convolution on the parameter defining the bias. Finally we mention that the Gibbs properties of Bernoulli convolutions have been studied only for some special cases, see [17], but the Gibbs properties of hidden Markov measures are quite well understood [5, 20, 13]. It would be interesting to see whether this theory for hidden Markov measures could be more widely applied to Bernoulli convolutions.

Acknowledgements

Many thanks to Karma Dajani, Nikita Sidorov, Simon Baker and Vaughn Climenhaga for many interesting and helpful discussions. This work was supported by the Dutch Organisation for Scientific Research (NWO) grant number 613.001.022 and the EPSRC.

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